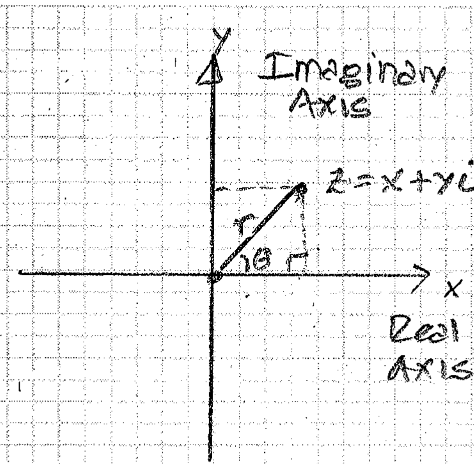
Complex Numbers. Part II



magnitude or modulus = $|Z|$

$$|Z| = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z = x + yi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\cdot\sin\theta)$$

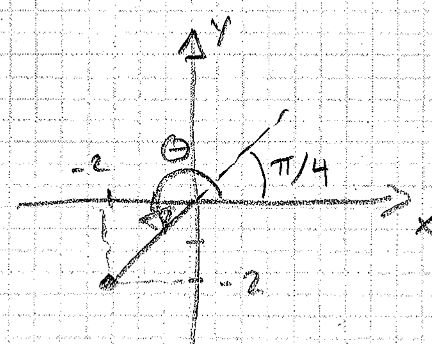
Rectangular Form — Polar Form

θ = argument

$0 \le \theta \le 2\pi$

Exercises

1) Plot the complex number $-2-2i$ and write it in polar form



$$r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(-2)^2}$$

$$= \sqrt{8} = 2\sqrt{2}$$

$$\cancel{\theta} = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1) = \pi/4$$

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

$$-2-2i = r(\cos\theta + i\sin\theta) = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Multiplication and Division of Complex Numbers in Polar Form

$$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

Example Given $z_1 = 2(\cos 10^\circ + i\sin 10^\circ)$

$z_2 = 3(\cos 60^\circ + i\sin 60^\circ)$

Find $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$

$$z_1 \cdot z_2 = 2\cdot 3\,(\cos(10^\circ+60^\circ) + i\sin(10^\circ+60^\circ))$$

$$= 6(\cos 70^\circ + i\sin 70^\circ)$$

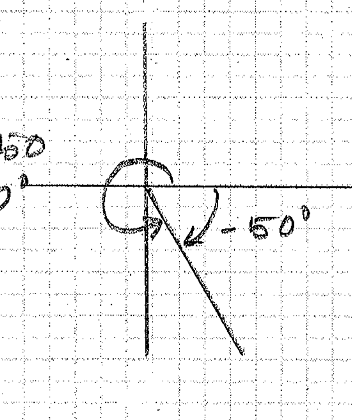
$$\frac{z_1}{z_2} = \frac{2}{3}(\cos(10^\circ-60^\circ) + i\sin(10^\circ-60^\circ))$$

$$= \frac{2}{3}(\cos(-50^\circ) + i\sin(-50^\circ))$$

$0 \le \theta \le 2\pi \qquad 0 \le \theta \le 360^\circ$

$$\frac{z_1}{z_2} = \frac{2}{3}(\cos 310^\circ + i\sin 310^\circ)$$

$360^\circ - 50$
$= 310^\circ$



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Powers of a Complex Number

$z = r\,(\cos\theta + i\sin\theta)$

DeMoivre's theorem

$z^n = r^n\,[\cos(n\theta) + i\sin(n\theta)]$

Example: $z = 2\,(\cos 30° + i\sin 30°)$

$z^3 = 2^3\,[\cos(3\cdot 30°) + i\sin(3\cdot 30°)]$

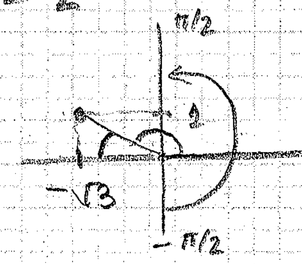
$= 8\,[\cos 90° + i\sin 90°]$

Example

$z = -\sqrt{3} + i \qquad z^4 = ?$

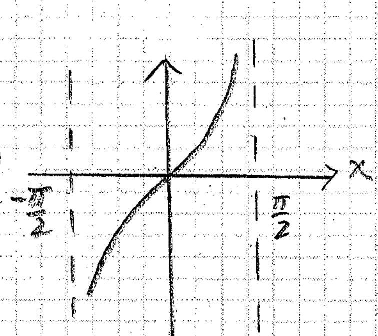
$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = ? \qquad \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$



$= -30°$

$\theta = 180° - 30° = 150°$



$z^4 = 2^4\,[\cos(4\cdot 150°) + i\sin(4\cdot 150°)]$

$= 16\,[\cos(600°) + i\sin(600°)]$

$= 16\,[\cos(240°) + i\sin(240°)]$

$$\begin{array}{r|l} & 1 \\ 360 & 600 \\ & \underline{360} \\ & 240° \end{array}$$

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Roots of a Complex Number

$$Z = r\,[\cos\theta + i\sin\theta] \qquad Z \neq 0$$

$n \geq 2$

$$\sqrt[n]{Z} = \sqrt[n]{r}\,[\cos\alpha + i\sin\alpha]$$

where $\alpha = \dfrac{\theta + 360k}{n}$ or $\alpha = \dfrac{\theta + 2k\pi}{n}$

for $k = 0, 1, \ldots, n-1$

Ex: $Z = -\sqrt{3} + i \qquad \sqrt[3]{Z} = ?$

$n = 3$

$$Z = 2\,[\cos 150^\circ + i\sin 150^\circ]$$

$k=0$ $\quad Z_1 = \sqrt[3]{2}\left[\cos\left(\dfrac{150^\circ + 360 * 0}{3}\right) + i\sin\left(\dfrac{150^\circ + 360 * 0}{3}\right)\right]$

$$Z_1 = \sqrt[3]{2}\,[\cos 50^\circ + i\sin 50^\circ]$$

$k=1$ $\quad Z_2 = \sqrt[3]{2}\left[\cos\left(\dfrac{150 + 360 * 1}{3}\right) + i\sin\left(\dfrac{150 + 360 * 1}{3}\right)\right]$

$$= \sqrt[3]{2}\,[\cos 170^\circ + i\sin 170^\circ]$$

$k=2$ $\quad Z_3 = \sqrt[3]{2}\left[\cos\left(\dfrac{150 + 360 * 2}{3}\right) + i\sin\left(\dfrac{150 + 360 * 2}{3}\right)\right]$

$$= \sqrt[3]{2}\,[\cos(290^\circ) + i\sin(290^\circ)]$$