DENSITIES AND DISTRIBUTIONS: DISCRETE VS CONTINUOUS

Probability distribution of X
Density function of X
Distribution function of X

Discrete Variables

If X is a discrete random variable, the function

$$f(x) = P(X = x)$$

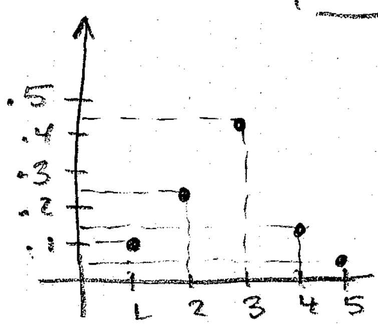
for each x in X is called the probability distribution of X if

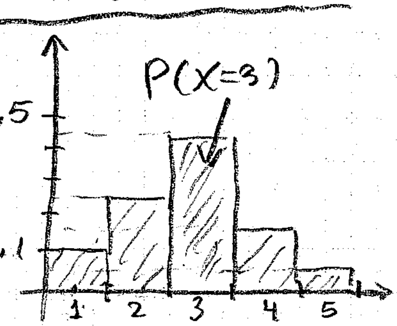
i) $f(x) \geq 0$ for all x in X

ii) $\sum f(x) = 1$

Example

X	1	2	3	4	5
$f(x)$	.10	.25	.45	.15	.05





Probability Histogram

Continuous Variables

If X is a continuous random variable, the function $f(x)$ is called its probability density function if

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

for any real constants a and b such that $a \leq b$ and

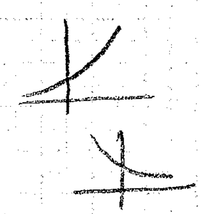
i) $f(x) \geq 0$ for $-\infty < x < \infty$

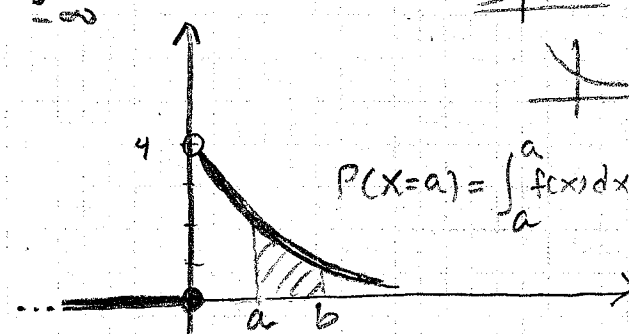
ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Example

$$f(x) = \begin{cases} 4e^{-4x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{\infty} 4e^{-4x}\,dx = 1$$





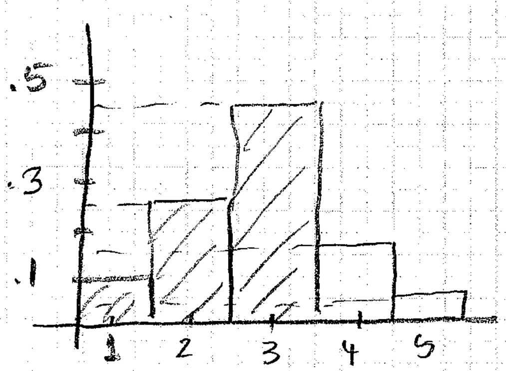
$$P(X=a) = \int_a^a f(x)\,dx = 0$$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Distribution Function

$$F(a) = P(X \le a) = \sum_{x \le a} f(x)$$

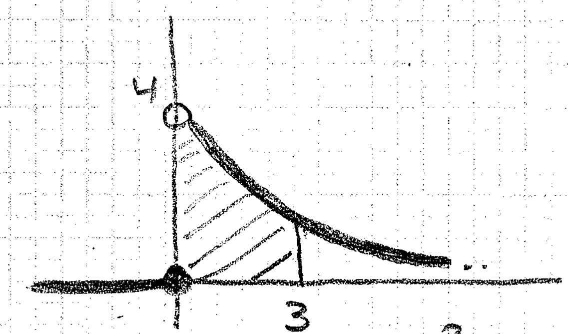
Ex:



$$F(3) = P(X \le 3) = \sum_{x \le 3} f(x)$$

Distribution Function

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$



$$F(3) = P(X \le 3) = \int_{-\infty}^{3} 4e^{-4x}\,dx$$

$$= \int_{0}^{3} 4e^{-4x}\,dx$$

To prove that $f(x) = \begin{cases} 4e^{-4x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$ is a probability density function, we have to prove that

$$\int_{-\infty}^{\infty} 4e^{-4x}\,dx = 1.$$

$$\int_{\infty}^{\infty} 4e^{-4x}\,dx = \int_{0}^{\infty} 4e^{-4x}\,dx = \lim_{t \to \infty} \int_{0}^{t} 4e^{-4x}\,dx = \lim_{t \to \infty} \left[\frac{4e^{-4x}}{-4}\right]_0^t$$

$$= \lim_{t \to \infty} \left[-\frac{1}{e^{4x}}\right]_0^t = \lim_{t \to \infty} \left[-\frac{1}{e^{4t}} - \left(-\frac{1}{e^{4\cdot 0}}\right)\right]$$

$$= \lim_{t \to \infty} \left[1 - \frac{1}{e^{4t}}\right] = 1$$

$$F(3) = \int_{0}^{3} 4e^{-4x}\,dx = 4 \cdot \frac{e^{-4x}}{-4}\Big|_0^3 = -\frac{1}{e^{4x}}\Big|_0^3 = \left(-\frac{1}{e^{12}} - \left(-\frac{1}{e^{0}}\right)\right)$$

$$= 1 - \frac{1}{e^{12}} \approx .999993856$$