# INFINITE SEQUENCES

Informally, an infinite sequence is an unending list of numbers $a_1, a_2, a_3, \ldots, a_{n-1}, a_n, \ldots$

Notation: $\{a_1, a_2, a_3, \ldots, a_{n-1}, a_n, \ldots\} = \{a_n\}_{n=1}^{\infty}$

Examples

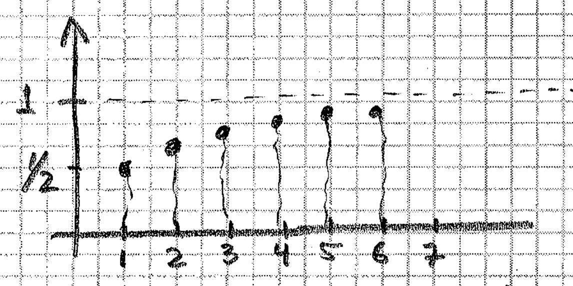
1) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

2) $\left\{\sqrt{n-2}\right\}_{n=2}^{\infty} = \left\{0, 1, \sqrt{2}, \ldots\right\}$

**Formally**, a sequence is a function whose domain is a set of integers $f(n) = a_n$ $n = 1, 2, 3, \ldots$

Since sequences are functions, we can speak of the limit of a sequence.

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$




A sequence $\{a_n\}$ has a limit $L$, $\lim_{n\to\infty} a_n = L$, if we can make $a_n$ as close to $L$ as we want by making $n$ sufficiently large.

If the limit exists, we say the sequence converges to $L$; otherwise, we say that it diverges

# PROPERTIES OF LIMITS OF SEQUENCES

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

1) $\lim\limits_{n\to\infty} c = c$

2) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

3) $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$

4) $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot \lim\limits_{n\to\infty} a_n$

5) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

6) $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0$

7) $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

8) If the sequence of even-numbered terms converges to L and the sequence of odd-numbered terms converges to L, then the sequence converges to L

Example: $\left\{(-1)^n \cdot \dfrac{1}{n}\right\}_{n=1}^{\infty} \longrightarrow 0$

for even values of n: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots \longrightarrow 0$

" odd " " : $-1, -\frac{1}{3}, -\frac{1}{5}, \ldots \longrightarrow 0$

Example: Determine whether the sequence converges

$$\left\{ \frac{3n}{2n+5} \right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \frac{3n}{2n+5} = \lim_{n\to\infty} \frac{\frac{3n}{n}}{\frac{2n}{n}+\frac{5}{n}} = \lim_{n\to\infty} \frac{3}{2+\frac{5}{n}} = \frac{3}{2}$$

Example Determine if the sequence converges

$$\left\{ n^2 e^{-n} \right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} n^2 e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2}{e^x} = 0$$