INFINITE SEQUENCES. PART II

The Squeezing theorem for sequences.

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences such that

$$a_n \leq b_n \leq c_n \quad \text{for all values of } n > M$$

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

Example: Determine if the sequence $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$ converges.

$$-1 \leq \sin n \leq 1$$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$ $\therefore \lim_{n\to\infty} \frac{\sin n}{n} = 0$

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theorem. If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

Example: Determine if the sequence $\left\{\frac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{\infty}$ converges

$$|a_n| = \left|\frac{(-1)^{n+1}}{n^2}\right| = \frac{1}{n^2} \qquad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$\therefore \lim_{n\to\infty} a_n = 0$ $\therefore$ Sequence converges to zero

MONOTONICITY A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be

strictly increasing if $a_1 < a_2 < \cdots < a_n < \cdots$

increasing if $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$

strictly decreasing if $a_1 > a_2 > \cdots > a_n > \cdots$

decreasing if $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$

A sequence is monotone if it is increasing or decreasing.
A sequence is strictly monotone if it is strictly increasing or strictly decreasing.

TESTS FOR MONOTONICITY $f(n) = a_n$

$a_{n+1} - a_n > 0$	$a_{n+1}/a_n > 1$	$f'(x) > 0$	strictly increasing
$a_{n+1} - a_n < 0$	$a_{n+1}/a_n < 1$	$f'(x) < 0$	strictly decreasing
$a_{n+1} - a_n \geq 0$	$a_{n+1}/a_n \geq 1$	$f'(x) \geq 0$	increasing
$a_{n+1} - a_n \leq 0$	$a_{n+1}/a_n \leq 1$	$f'(x) \leq 0$	decreasing

Example: Determine if the sequence is monotone $\left\{ \frac{3^n}{(2n)!} \right\}_{n=1}^{\infty}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{[2(n+1)]!}}{\frac{3^n}{(2n)!}} = \frac{3^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{3^n} = \frac{\cancel{3^n} \cdot 3 \quad \cancel{(2n)!}}{(2n+2)(2n+1) \cdot \cancel{(2n)!} \, \cancel{3^n}}$$

$$= \frac{3}{(2n+2)(2n+1)} < 1$$ the sequence is strictly decreasing