HOW TO CALCULATE THE INVERSE OF A SQUARE MATRIX

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} such that

$$A \times A^{-1} = I_n$$

and $A^{-1} \times A = I_n$

then A^{-1} is called the inverse of A.

If the matrix A has an inverse, it is called non-singular

If the matrix A does not have an inverse, it is called singular.

Example: Find the inverse of the given matrix, if it exists

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

1) Form the matrix $[A \mid I_3]$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

2) Apply row operations to transform $[A \mid I_3]$ into reduced row echelon form

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \square & \square & \square \\ 0 & 1 & 0 & \square & \square & \square \\ 0 & 0 & 1 & \square & \square & \square \end{array}\right]$$

$R_1 \times (1) + R_2 \rightarrow R_2$

$$\begin{aligned} 1 \times (1) + (-1) &= 0 \\ 0 \times (1) + 1 &= 1 \\ 1 \times (1) + (-1) &= 0 \\ 1 \times (1) + 0 &= 1 \\ 0 \times (1) + 1 &= 1 \\ 0 \times (1) + 0 &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \times (-1) + R_3 \rightarrow R_3$

$$\begin{aligned} 1 \times (-1) + 1 &= 0 \\ 0 \times (-1) + 2 &= 2 \\ 1 \times (-1) + 3 &= 2 \\ 1 \times (-1) + 0 &= -1 \\ 0 \times (-1) + 0 &= 0 \\ 0 \times (-1) + 1 &= 1 \end{aligned} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \times (-2) + R_3 \rightarrow R_3$

$$\begin{aligned} 0 \times (-2) + 0 &= 0 \\ 1 \times (-2) + 2 &= 0 \\ 0 \times (-2) + 2 &= 2 \\ 1 \times (-2) + (-1) &= -3 \\ 1 \times (-2) + 0 &= -2 \\ 0 \times (-2) + 1 &= 1 \end{aligned} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 & -2 & 1 \end{array}\right]$$

$R_3 \times (1/2) \rightarrow R_3$

$$\begin{aligned} 0 \times (1/2) &= 0 \\ 0 \times (1/2) &= 0 \\ 2 \times (1/2) &= 1 \\ -3 \times (1/2) &= -3/2 \\ -2 \times (1/2) &= -1 \\ 1 \times (1/2) &= 1/2 \end{aligned} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3/2 & -1 & 1/2 \end{array}\right]$$

$R_3 \times (-1) + R_1 \to R_1$

$0 \times (-1) + 1 = 1$
$0 \times (-1) + 0 = 0$
$1 \times (-1) + 1 = 0$
$-3/2 \times (-1) + 1 = 5/2$
$-1 \times (-1) + 0 = 1$
$1/2 \times (-1) + 0 = -1/2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & 1 & -1/2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3/2 & -1 & 1/2 \end{array}\right]$$
$$\underbrace{\qquad\qquad}_{I_3} \qquad \underbrace{\qquad\qquad}_{A^{-1}}$$

CHECK

$A \times A^{-1} = I_3$ and $A^{-1} \times A = I_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 5/2 & 1 & -1/2 \\ 1 & 1 & 0 \\ -3/2 & -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5/2 & 1 & -1/2 \\ 1 & 1 & 0 \\ -3/2 & -1 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2 Find A^{-1} if it exists

singular

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \square & \square & \square \\ 0 & 1 & 2 & \square & \square & \square \\ 0 & 0 & 0 & \square & \square & \square \end{array}\right]$$