POLYNOMIAL FUNCTIONS. PART II

Factor theorem:

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function.

1) If $f(c)=0$ then $x-c$ is a factor of $f(x)$, and
2) If $x-c$ is a factor of $f(x)$ then $f(c)=0$

Ex: $f(x) = x^3 - 2x^2 - 5x + 6$

$f(1) = 1^3 - 2\cdot 1^2 - 5\cdot 1 + 6 = 1 - 2 - 5 + 6 = 0$

therefore $(x-1)$ is a factor of $f(x)$

Synthetic Division

$$\begin{array}{r|rrrr} & 1 & -2 & -5 & 6 \\ 1 & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$\underline{f(x) = (x-1)(x^2-x-6)}$

theorem:

If $f(x)$ is a polynomial of degree n, then it has at most n distinct real zeros.

Rational Zeros theorem

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
be a polynomial where $n \geq 1$, $a_n \neq 0$, $a_0 \neq 0$
and all coefficients are integers.

If $\frac{p}{q}$ is a zero of $f(x)$ then

p is a factor of a_0, and
q is a factor of a_n.

Example

$$f(x) = 3x^4 - 5x^3 + x^2 - 5x - 2$$

$\frac{p}{q}$ p factor of -2 ± 1 ± 2
q " " 3 ± 1 ± 3

possible zeros (rational) $1, 2, -1, -2, 1/3, -1/3, 2/3, -2/3$

How do we test?
a) using the factor theorem, or
b) " synthetic division

test $x = 1$ $f(1) = 3 \cdot 1^4 - 5 \cdot 1^3 + 1^2 - 5 \cdot 1 - 2 \neq 0$ not a zero

$$\begin{array}{r|rrrrr} & 3 & -5 & 1 & -5 & -2 \\ 1 & & 3 & -2 & -1 & -6 \\ \hline & 3 & -2 & -1 & -6 & \neq 0 \end{array}$$

$f(2) = 3 \cdot 2^4 - 5 \cdot 2^3 + 2^2 - 5 \cdot 2 - 2 = 0$ ✓

$x = 2$ is a zero.

$$\begin{array}{r|rrrrr} & 3 & -5 & 1 & -5 & -2 \\ 2 & & 6 & 2 & 6 & 2 \\ \hline & 3 & 1 & 3 & 1 & 0 \end{array}$$

← 4th degree

reduced polynomial 3rd degree

$f(x) = (x-2)(3x^3 + x^2 + 3x + 1)$

$$\begin{array}{r|rrrr} & 3 & 1 & 3 & 1 \\ -1/3 & & -1 & 0 & -1 \\ \hline & 3 & 0 & 3 & 0 \end{array}$$

$\cancel{1}$, $\bar{2}$, -1, -2, $1/3$

$-1/3$, $2/3$, $-2/3$

$x = -1/3$ is a zero $\therefore$ $x - 1/3$ is a factor

$f(x) = (x-2)(x-1/3)\underbrace{(3x^2+3)}_{\text{irreducible}}$

$3x^2 + 3 = 0 \Rightarrow 3x^2 = -3 \Rightarrow \boxed{x^2 = -1}$

$f(x) = (x-2)(x-1/3)\,3(x^2+1)$

$= (x-2)(3x-1)(x^2+1)$

Real zeros: 2, $-1/3$