THE FUNDAMENTAL THEOREM OF ALGEBRA AND THE CONJUGATE PAIRS THEOREM.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Fundamental theorem of Algebra

Every complex polynomial $f(x)$ of degree $n \geq 1$ has at least one complex zero

The Factor theorem for Complex Polynomials

Every complex polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has exactly n zeros $r_1, r_2, \ldots, r_n$ and it can be factored as

$$f(x) = a_n (x - r_1)(x - r_2) \cdots (x - r_n)$$

The Conjugate Pair theorem

If $f(x)$ is a polynomial with real coefficients and $a+bi$ is a zero, then its conjugate $a-bi$ is also a zero.

Example: A polynomial of degree 5 with real coefficients has the zeros 2, $3+i$, and $-4i$. Find the remaining zeros.

Since $3+i$ is a zero, so is $3-i$
" $-4i$ " , " $4i$

Example: Find a polynomial of degree 3 with real coefficients that has zeros 2 and $1-i$

$1+i$ is also a zero
If we choose the leading coefficient $a_3 = 1$ then

$$f(x) = 1(x-2)(x-(1-i))(x-(1+i))$$
$$= (x-2)(x-1+i)(x-1-i)$$
$$= (x-2)(x^2 - x - \cancel{ix} - x + 1 + \cancel{i} + \cancel{ix} - \cancel{i} - i^2)$$
$$= (x-2)(x^2-2x+2)$$
$$= x^3 - 2x^2 + 2x - 2x^2 + 4x - 4$$
$$= x^3 - 4x^2 + 6x - 4$$

Example: Find the complex zeros of the polynomial

$$f(x) = 3x^4 - 7x^3 + 14x^2 - 28x + 8$$

$\frac{p}{q}$ p is a factor of 8 : ± 1 ± 2 ± 4 ± 8

q " " " 3 : ± 1 ± 3

Descartes's Rule of Signs

		3	-7	14	-28	8
~~4~~	2		6	-2	24	-8
		3	-1	12	-4	0
~~4~~ ~~8~~	1/3		1	0	4	
		3	0	12	0	

$3x^2 + 12 = 0$ $a = 3$ $b = 0$ $c = 12$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{-4*3*12}}{2*3} = \frac{\pm\sqrt{-144}}{6}$$

$$= \pm\frac{12i}{6} = \pm 2i$$

Complex Roots: $2, 1/3, 2i, -2i$