TRIGONOMETRIC INTEGRALS. PART I

Integrating Powers of Sine and Cosine

$$I = \int \sin^2 x \, dx$$

$$\boxed{\sin^2 x = \frac{1}{2}(1-\cos(2x))} \qquad \boxed{\cos^2 x = \frac{1}{2}(1+\cos(2x))}$$

$$I = \int \frac{1}{2}(1-\cos(2x))\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx$$

$u = 2x$
$du = 2\,dx$

$$= \frac{1}{4}\int (1-\cos(2x))\,2\,dx$$

$$= \frac{1}{4}\int (1-\cos u)\,du$$

$$= \frac{1}{4}\int du - \frac{1}{4}\int \cos u\,du = \frac{1}{4}u - \frac{1}{4}\sin u + C$$

$$= \frac{1}{4}2x - \frac{1}{4}\sin(2x) + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\underline{I = \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C}$$

$$I = \int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx.$$

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$= \int (1-\cos^2 x)\sin x\,dx = -\int (1-\cos^2 x)(-\sin x\,dx)$$

$$\left.\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}\right\} = -\int (1-u^2)\,du$$

$$= -\int du + \int u^2\,du = -u + \frac{u^3}{3} + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C = \frac{1}{3}\cos^3 x - \cos x + C$$

Similarly, we get $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$

$$I = \int \sin^4 x\,dx = \int \sin^2 x \sin^2 x\,dx$$

$$= \int \tfrac{1}{2}(1-\cos(2x))\,\tfrac{1}{2}(1-\cos(2x))\,dx$$

$$= \frac{1}{4}\int (1-\cos(2x))^2\,dx = \frac{1}{4}\int [1 - 2\cos(2x) + \cos^2(2x)]\,dx$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$I = \frac{1}{4}\int dx - \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx$$

$u = 2x$
$du = 2\,dx$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int \cos(2x)\,2\,dx + \frac{1}{8}\int \cos^2(2x)\,2\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int \cos u\,du + \frac{1}{8}\int \cos^2 u\,du$$

$$= \frac{1}{4}x - \frac{1}{4}\sin u + \frac{1}{8}\left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right] + C$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{16}2x + \frac{1}{32}\sin(4x) + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

Reduction Formulas