TRIGONOMETRIC INTEGRALS. PART III

Integrals of Products of Sine and Cosine

$$\int \sin^m x \cdot \cos^n x \, dx$$

Ex 1: $\int \sin^4 x \cos^5 x \, dx$

Ex 2: $\int \sin^5 x \cos^4 x \, dx$

Ex 3: $\int \sin^4 x \cdot \cos^4 x \, dx$

Ex 1: $I = \int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cdot \cos x \, dx$

NOTE: $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$\Rightarrow \cos^4 x = (\cos^2 x)^2 = (1 - \sin^2 x)^2$

$$I = \int \sin^4 x \, (1 - \sin^2 x)^2 \cdot \underbrace{\cos x \, dx}_{du}$$

$u = \sin x$

$du = \cos x \, dx$

$(a-b)^2 = a^2 - 2ab + b^2$

$$I = \int u^4 (1 - u^2)^2 \, du = \int u^4 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^4 - 2u^6 + u^8) \, du = \frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C$$

EX 2: $I = \int \sin^5 x \cos^4 x \, dx = \int \sin^4 x \cos^4 x \sin x \, dx$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x \Rightarrow$

$\Rightarrow \sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2$

$I = \int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx = $ etc...

$u = \cos x$

$du = -\sin x \, dx$

EX 3: $I = \int \sin^4 x \cos^4 x \, dx$

NOTE: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

$$I = \int \left[\frac{1}{2}(1-\cos(2x))\right]^2 \left[\frac{1}{2}(1+\cos(2x))\right]^2 dx$$

$$= \frac{1}{16} \int \left[(1-\cos(2x))(1+\cos(2x))\right]^2 dx$$

NOTE: $(a-b)(a+b) = a^2 - b^2$

$$= \frac{1}{16} \int (1 - \cos^2(2x))^2 \, dx$$

NOTE: $1 - \cos^2(2x) = \sin^2(2x)$

$$= \frac{1}{16} \int \sin^4(2x) \, dx = \frac{1}{16} \frac{1}{2} \int \sin^4(2x) \cdot \underbrace{2 \cdot dx}_{du}$$

$u = 2x$

$du = 2 \, dx$

$$= \frac{1}{32} \int \sin^4 u \, du$$

$$= \frac{1}{32}\left(-\frac{1}{4}\sin^3 u \cos u - \frac{3}{8}\sin u \cos u + \frac{3}{8} u\right) + C$$